

Effective description of the short-time dynamics in open quantum systems

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We address the dynamics of a bosonic system coupled to either a bosonic or a magnetic environment, and derive a set of sufficient conditions that allow one to describe the dynamics in terms of the effective interaction with a classical fluctuating field. We find that for short interaction times the dynamics of the open system is described by a Gaussian noise map for several different interaction models and independently on the temperature of the environment. More generally, our results indicate that quantum environments may be described by classical fields whenever global symmetries lead to the definition of environmental operators that remain well defined when increasing the size of the environment.

I. INTRODUCTION

The modeling of any open quantum system (OQS) inherently implies that of its surroundings. However, knowing the quantum structure of the total Hamiltonian, including the details of the couplings between the principal system A and its environment Ξ , does not usually suffice to develop a simple and meaningful model of the overall system, due to Ξ being made of a very large number N of quantum components, a fact that we will hereafter take as integral to the definition of *environment*. On the other hand, knowing specific features of Ξ may help selecting a suitable formalism and/or some appropriate approximations, so as to devise the most effective strategies for tackling problems that cannot be otherwise studied.

As a matter of fact, the modeling of an effective description of Ξ and of its influence on A usually stems from intuitive and phenomenological arguments [1], or even from an arbitrary choice, rather than a formal derivation. One of the reasons why this is so typical in the study of OQS is that the large- N theories that have been extensively developed and used in quantum-field-theory since the 1970s (comprehensive bibliographies and discussions can be found for instance in Refs. [2, 3]) are not trivially applicable when the large- N system is not isolated, but rather coupled with a small, invariably quantum, principal system. Unless one decides that the latter is not “principal” at all, and can be hence neglected, several foundational issues arise in this setting, due to the difficult co-existence of quantum and classical formalisms, possibly made worse by the presence of thermal baths or stochastic agents.

Having this issue in mind, here we analyze a specific situation where a principal quantum system A interacts with an equally quantum environment Ξ , which is put into contact with a further external system T. If Ξ is macroscopic and T is a thermal bath at high temperature, it may appear intuitive, and naively understood, that A effectively evolves as if it were under the influence of a classical fluctuating field. This state-

ment, however, has the nature of an ansatz as far as it is not formally inferred, and conditions ensuring its validity are not given.

Several OQS have been indeed investigated to assess whether an effective description is viable, where the effects of the environment are described in terms of the coupling with a classical fluctuating field [4–16]. As a matter of fact, full equivalence has been shown only for single-qubit dephasing dynamics [9], with an explicit construction of the corresponding classical stochastic process. General arguments valid also for bipartite systems have been discussed [17–19] and the effects of the interaction with a classical field have been investigated in detail [20–27]. Parametric representation have also been used to show that classical variables can emerge in quantum Hamiltonians as environmental degrees of freedom [28–33].

In this work we go beyond pure dephasing and scrutinize the general idea that the dynamics of a quantum system with a macroscopic environment may be effectively described by a non-autonomous, i.e. time-dependent, Hamiltonian acting on the principal system only. In particular, we critically inspect the conditions for the validity of this hypothesis as a tool to understand whether it stems from Ξ being macroscopic, or the temperature being high, or from enforcing some other specific condition.

To this aim, we start, in Sec. II, by considering the case where A is a bosonic mode coupled with an equally bosonic environment, hereafter called B, which is made of N distinguishable modes that do not interact amongst themselves. The Hamiltonian reads

$$H = \nu a^\dagger a + \sum_k^N (\lambda_{1k} a^\dagger + \lambda_{2k} a) b_k + \sum_k^N (\lambda_{1k}^* a + \lambda_{2k}^* a^\dagger) b_k^\dagger + \sum_k^N \omega_k b_k^\dagger b_k \quad (1)$$

where $[a, a^\dagger] = 1$ and $[b_k, b_{k'}^\dagger] = \delta_{kk'}$, with $\nu, \omega \in \mathbb{R}$ and $\lambda_{1k}, \lambda_{2k} \in \mathbb{C}$, $\forall k$. Also, we have set $\hbar = 1$, as done throughout this work. Studying the evolution of the reduced density matrix for the principal system, we show that the short-time

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dynamics defined by Eq. (1) with $\omega_k \sim \omega \forall k$, can be described by an effective Hamiltonian acting on A only, $H_A^{\text{eff}}(\zeta)$, where the functions ζ embody the remnants of B in the form of classical, possibly fluctuating fields, depending on external parameters such as time and temperature.

The above model has a sibling that describes the case of a spin environment, hereafter called S, made by N distinguishable spin- $\frac{1}{2}$ particles that do not interact amongst themselves. Its dynamics is studied in Sec. III, as described by the Hamiltonian

$$H = \nu a^\dagger a + \sum_i^N (g_{1i} a^\dagger + g_{2i} a) \sigma_i^- + \sum_i^N (g_{1i}^* a + g_{2i}^* a^\dagger) \sigma_i^+ + \sum_i^N f_i \sigma_i^z \quad (2)$$

where $[\sigma_i^+, \sigma_{i'}^-] = 2\delta_{ii'} \sigma_i^z$, $[\sigma_i^z, \sigma_{i'}^\pm] = \pm \delta_{ii'} \sigma_i^\pm$, $f_i \in \mathbb{R}$ and $g_{1i}, g_{2i} \in \mathbb{C}$, $\forall i$. Despite differences with the case of a bosonic environment emerge, essentially due to the specific algebra of the spin operators, the short-time dynamics of this model for $f_i \sim f \forall i$ is also found to be properly described by an effective Hamiltonian $H_A^{\text{eff}}(\zeta)$.

Upon inspecting the dynamics of both systems in order to retrace the derivation of the short-time dynamics, we notice that no explicit condition on the value of N is involved. This is somehow surprising, given that B and S are named *environment* insofar as the number N of their quantum components is large, virtually infinite in the case of a macroscopic environment. Therefore, in order to understand whether a relation exists between a large value of N and the assumptions of short-time and almost flat spectrum used in Secs. II-III, in Sec. IV we take on the model (1) from a more abstract viewpoint. More specifically: we generalize the well-established procedures for deriving classical theories as large- N limit of quantum ones [34] to the case of composite quantum systems, and find that replacing quantum operators by classical fields for $N \rightarrow \infty$ requires that environmental operators stay well defined in such limit which, in turn, implies the environment to feature some global symmetry. In particular, we show that the renormalization of the couplings, which is necessary for the $N \rightarrow \infty$ limit to stay physically meaningful, reflects upon the short-time condition previously used. Also, we will discuss how a flat environmental spectrum is the key feature that guarantees the existence of a global symmetry in the theory defined by Eq. (1), namely the symmetry under permutation of different modes.

Overall, collecting our diverse results, we put forward the conjecture that non-autonomous Hamiltonians for closed quantum systems describe the short-time dynamics of interacting models involving at least one macroscopic subsystem. We also comment upon the symmetry properties allowing this subsystem to emerge as a macroscopic one, and the related features of its energy spectrum. Finally, we discuss the role of such symmetry properties in the design of a general procedure for deriving an effective non-autonomous Hamiltonian from an interacting microscopic model.

Sec. V closes the paper with some concluding remarks.

II. BOSONIC ENVIRONMENT

We consider the Hamiltonian (1), for either 1) $\lambda_{2k} = 0$, with $\lambda_k \equiv \lambda_{1k}$ finite, or 2) $\lambda_{1k} = 0$, with $\lambda_k \equiv \lambda_{2k}$ finite, $\forall k$, i.e.

$$H_1 = \nu a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + \sum_k \left(\lambda_k^* a b_k^\dagger + \lambda_k a^\dagger b_k \right), \quad (3)$$

$$H_2 = \nu a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + \sum_k \left(\lambda_k^* a^\dagger b_k^\dagger + \lambda_k a b_k \right). \quad (4)$$

These two distinct models are sometimes referred to as the linear-exchange and parametric-hopping ones: they essentially describe a dissipation (D) or an amplification (A) process, and will be hereafter labeled by an index $j = 1, 2$ representing the D,A-case, respectively. The Heisenberg equations of motion (EOM) for the mode operators are

$$\text{D-case:} \quad \dot{a} = i[H_1, a] = -i\nu a - i \sum_k \lambda_k b_k, \quad (5a)$$

$$\dot{b}_k = i[H_1, b_k] = -i\omega_k b_k - i\lambda_k^* a, \quad (5b)$$

$$\text{A-case:} \quad \dot{a} = i[H_2, a] = -i\nu a - i \sum_k \lambda_k^* b_k^\dagger, \quad (6a)$$

$$\dot{b}_k^\dagger = i[H_2, b_k^\dagger] = i\omega_k b_k^\dagger + i\lambda_k a. \quad (6b)$$

If the spectrum of the environment is narrow enough to write $\omega_k \simeq \omega \forall k$, the above EOM can be written as

$$\text{D-case:} \quad \dot{a} = -i\nu a - i\Lambda b, \quad \dot{b} = -i\omega b - i\Lambda a, \quad (7)$$

$$\text{A-case:} \quad \dot{a} = -i\nu a - i\Lambda b^\dagger, \quad \dot{b}^\dagger = i\omega b^\dagger + i\Lambda a, \quad (8)$$

where the bosonic operator b is defined as

$$b \equiv \frac{1}{\Lambda} \sum_k \lambda_k b_k, \quad \text{with} \quad \Lambda^2 \equiv \sum_k |\lambda_k|^2. \quad (9)$$

The above Eqs. (7-8) are the same EOM that one would obtain starting from the two-mode bosonic Hamiltonians

$$\text{D-case:} \quad \nu a^\dagger a + \omega b^\dagger b + \Lambda(ab^\dagger + a^\dagger b), \quad (10)$$

$$\text{A-case:} \quad \nu a^\dagger a + \omega b^\dagger b + \Lambda(a^\dagger b^\dagger + ab), \quad (11)$$

describing two oscillators, with different frequencies ν and ω , exchanging quanta through a linear interaction. Notice, though, that such direct relation only exists in the case of an almost-flat spectrum, $\omega_k \sim \omega, \forall k$.

Both systems of Eqs. (7) and (8) can be solved by Laplace transform, using the rule $\dot{a}(s) = s\tilde{a}(s) - a(0)$ to obtain algebraic equations from differential ones. Few calculations lead us, after back-transforming and recalling that the index $j = 1, 2$ refers to the D,A-case respectively, to the solutions

$$a(t) = e^{-iH_j t} a e^{iH_j t} = [\mu_j(t) a + \pi_j(t) \mathcal{B}_j] e^{-i\omega_j t}, \\ \mathcal{B}_j(t) = e^{-iH_j t} \mathcal{B}_j e^{iH_j t} = [(-)^j \pi_j^*(t) a + \mu_j^*(t) \mathcal{B}_j] e^{-i\omega_j t}, \quad (12)$$

where $\mathcal{B}_1 = b$, $\mathcal{B}_2 = b^\dagger$,

$$\mu_j(t) = \cos(\Delta_j t) - i \frac{\delta_j}{\Delta_j} \sin(\Delta_j t), \quad (13a)$$

$$\pi_j(t) = -i \frac{\Lambda}{\Delta_j} \sin(\Delta_j t), \quad (13b)$$

with

$$\delta_j = \frac{1}{2} (\nu + (-)^j \omega), \quad (14a)$$

$$\omega_j = \frac{1}{2} (\nu - (-)^j \omega), \quad (14b)$$

$$\Delta_j^2 = |\delta_j^2 - (-)^j \Lambda^2|, \quad (14c)$$

and we have used $\mu_j^*(t) = \mu_j(-t)$. The overall phase factors in the rightmost terms of Eqs. (12) suggest that a natural interaction picture exists, corresponding to frames rotating at frequency ω_j . We will use these frames in the following, so as to omit those phase factors. Further notice that $|\mu_j(t)|^2 - (-)^j |\pi_j(t)|^2 = 1$, ensuring that $[a(t), a^\dagger(t)] = [b(t), b^\dagger(t)] = 1$, $\forall t$ and also that $|\mu_j(t)|^2 + (-)^j \pi_j^2(t) = 1$, meaning that the evolutions correspond to rotations in the rotating frames.

Our goal is now to obtain an effective Hamiltonian $H_A^{\text{eff}}(\zeta)$, acting on A only, without renouncing to the quantum character of its companion B. This means that we can consider nothing but the time dependence of the reduced density matrix for A

$$\rho_A(t) = \text{Tr}_B [e^{-iH_j t} \rho_A \otimes \rho_B e^{iH_j t}] \equiv \mathcal{E}_j[\rho_A](t), \quad (15)$$

with the notation $\rho_X \equiv \rho_X(0)$ used hereafter. In particular, as already implied by Eq. (15), we want to derive the explicit form of the dynamical map $\mathcal{E}_j[\rho_A]$ upon assuming that at $t = 0$ the system A+B is in a factorized state $\rho_A \otimes \rho_B$. Moreover, we specifically take B initially prepared in the state at thermal equilibrium

$$\rho_B = \frac{1}{1 + n_T} \left(\frac{n_T}{1 + n_T} \right)^{b^\dagger b}, \quad (16)$$

where $n_T = (e^{\omega/T} - 1)^{-1}$ is the thermal number of photons, and we have set the Boltzmann constant equal to 1.

After this choice, that implicitly means that B further interacts with a third system T, specifically a thermal bath due to the choice of the state in Eq. (16), we can positively move towards the derivation of the field ζ entering H_A^{eff} , and of its possible dependence on some external parameter. To this aim we first write the initial state of A+B using the Glauber formula,

$$\rho_A \otimes \rho_B = \iint \frac{d^2 \gamma' d^2 \gamma''}{\pi^2} \chi[\rho_A](\gamma') \chi[\rho_B](\gamma'') D_a^\dagger(\gamma') \otimes D_b^\dagger(\gamma''), \quad (17)$$

where $\chi[\rho](\gamma) = \text{Tr}[\rho D(\gamma)]$ is the characteristic function of the state ρ , and $D_x(\gamma) = \exp\{\gamma x^\dagger - \gamma^* x\}$, with $[x, x^\dagger] = 1$, is the bosonic displacement operator. In order to get the argument of the partial trace in Eq. (15), we use Eqs. (12)

to write the evolution of the displacement operators entering Eq. (17),

$$e^{-iH_j t} D_a^\dagger(\gamma') \otimes D_b^\dagger(\gamma'') e^{iH_j t} = D_a^\dagger[\mu_j^*(t)\gamma' + \pi_j^*(t)\gamma''] \otimes D_b^\dagger[\pi_j^*(t)\gamma' + \mu_j(t)\gamma'']. \quad (18)$$

We then perform the partial trace using $\text{Tr}[D(\gamma)] = \pi \delta^{(2)}(\gamma)$, so as to get

$$\begin{aligned} \mathcal{E}_j[\rho_A](t) &= \int \frac{d^2 \gamma'}{\pi} \chi[\rho_A](\gamma') \chi[\rho_B] \left(-\frac{\gamma' \pi_j^*(t)}{\mu_j(t)} \right) D^\dagger \left(\frac{\gamma'}{\mu_j(t)} \right) \\ &= \int \frac{d^2 \gamma}{\pi} |\mu_j(t)|^2 \chi[\rho_A](\gamma \mu_j(t)) \chi[\rho_B](-\gamma \pi_j^*(t)) D^\dagger(\gamma), \end{aligned} \quad (19)$$

where, in the last step, we made the substitution $\gamma' \rightarrow \gamma \mu_j(t)$.

Upon expanding the coefficients (13) for $\Delta_j t \ll 1$,

$$\mu_j(t) \simeq 1 - i\delta_j t + O(t^2), \quad (20a)$$

$$\pi_j(t) \simeq -i\Lambda t + O(t^2), \quad (20b)$$

$$|\mu_j(t)|^2 \simeq 1 + O(t^2), \quad (20c)$$

and using the explicit form of the characteristic function of the thermal state, $\chi[\rho_B](\gamma) = \exp\{-|\gamma|^2(n_T + \frac{1}{2})\}$, we finally write

$$\begin{aligned} \rho_A(t) &= \mathcal{E}_j[\rho_A](t) \\ &= \int \frac{d^2 \gamma}{\pi} \chi[\rho_A](\gamma) e^{-|\gamma|^2 \sigma^2(t)} D^\dagger(\gamma), \end{aligned} \quad (21)$$

with $\sigma^2(t) = \Lambda^2 t^2 (n_T + \frac{1}{2})$.

We now wonder whether the above map is realized by some known unitary evolution involving the interaction with a classical environment only. Indeed, by first noticing that for any state ϱ it is

$$\chi[\varrho](\gamma) e^{-|\gamma|^2 \sigma^2} = \chi[\varrho_{\text{GN}}](\gamma), \quad (22)$$

with

$$\varrho_{\text{GN}} \equiv \int \frac{d^2 \alpha}{\pi \sigma^2} e^{-\frac{|\alpha|^2}{\sigma^2}} D(\alpha) \varrho D^\dagger(\alpha), \quad (23)$$

we recognize in Eq. (23) the Kraus decomposition corresponding to a Gaussian noise (GN) channel, namely a random displacement with Gaussian modulated amplitude [35].

The same map [36, 37] describes the evolution of a bosonic system in the presence of a classical fluctuating field, i.e. governed by a non-autonomous Hamiltonian of the form

$$H_{\text{stoc}}(t) = \nu a^\dagger a + a \zeta^*(t) e^{i\omega_\zeta t} + a^\dagger \zeta(t) e^{-i\omega_\zeta t}, \quad (24)$$

where $\zeta(t)$ is a random classical field described by a Gaussian stochastic process $\zeta(t) = \zeta_x(t) + i\zeta_y(t)$ with zero mean

$[\zeta_x(t)]_\zeta = [\zeta_y(t)]_\zeta = 0$ and diagonal structure of the autocorrelation function

$$[\zeta_x(t_1)\zeta_x(t_2)]_\zeta = [\zeta_y(t_1)\zeta_y(t_2)]_\zeta = K(t_1, t_2), \quad (25a)$$

$$[\zeta_x(t_1)\zeta_y(t_2)]_\zeta = [\zeta_y(t_1)\zeta_x(t_2)]_\zeta = 0. \quad (25b)$$

The function $\sigma(t)$ in Eq. (22) is

$$\sigma(t) = \int_0^t \int_0^t dt_1 dt_2 \cos[\delta_\zeta(t_1 - t_2)] K(t_1, t_2), \quad (26)$$

where $\delta_\zeta = \omega_\zeta - \nu$ is the detuning between the natural frequency of A and the central frequency of the classical field $\zeta(t)$. The map (22) may be obtained, for instance, upon considering the classical environment fluctuating according to a Gaussian Ornstein-Uhlenbeck stochastic process [38] characterized by the autocorrelation function

$$K_\tau^{\text{ou}}(t_1 - t_2) = \frac{G}{2\tau} e^{-\frac{1}{\tau}|t_1 - t_2|}, \quad (27)$$

where τ is the correlation time, and G is the amplitude of the process. In the short-time limit, one easily finds that

$$\sigma(t) = \frac{G}{2\tau} t^2. \quad (28)$$

In conclusion, we have shown that, as far as $t \ll |\Delta_j|^{-1}$, the effective Hamiltonian $H_A^{\text{eff}}(\zeta(t))$ equals $H_{\text{stoc}}(t)$, meaning that

$$H_A^{\text{eff}}(\zeta(t)) = \nu a^\dagger a + a \zeta^*(t) e^{i\omega_\zeta t} + a^\dagger \zeta(t) e^{-i\omega_\zeta t}, \quad (29)$$

with the field $\zeta(t)$ as from Eqs. (25-28), and $G = 2\tau\Lambda^2(n_\tau + \frac{1}{2})$.

Notice that the dynamical map for A in the short-time limit, Eq. (21), is the same in the D- and A-case. However, due to the j dependence of Δ_j , the condition defining the above short-time limit is different in the two cases. In fact, the difference is removed when the number of environmental modes becomes large, and the effective coupling $\Lambda = \sqrt{\sum_k \lambda_k^2}$ increases accordingly, so that

$$t \ll \frac{1}{\sqrt{(\nu \mp \omega)^2 \pm \Lambda^2}} \xrightarrow{\text{large-}N} t \ll \frac{1}{\Lambda}, \quad (30)$$

which establishes a relation between the short-time constraint and some large- N condition that will be further discussed later on.

Overall, we have that the interaction (either exchange or hopping) of an oscillator with a bosonic environment induces a dynamics that is amenable to a description in terms of the interaction with a fluctuating classical field if the following conditions can be, at least approximately, met:

- (i) flat environmental energy-spectrum
- (ii) short interacting times
- (iii) environment at thermal equilibrium.

It is worth noticing that, if conditions (i)-(iii) hold, the above description in terms of classical fields is valid at all temperatures.

III. MAGNETIC ENVIRONMENT

We now consider the situation described by the Hamiltonian (2), i.e. that of a bosonic mode A interacting linearly with a magnetic system S, made of N spin- $\frac{1}{2}$ particles, each described by its respective Pauli matrices $(\sigma_i^x, \sigma_i^y, \sigma_i^z) \equiv \boldsymbol{\sigma}_i$. As in Sec. II, we consider both the dissipation and the amplification case. Setting 1) $g_{2i} = 0$, with $g_i \equiv g_{1i}$ finite, and 2) $g_{1i} = 0$, with $g_i \equiv g_{2i}$ finite, $\forall i$, from Eq. (2) we get

$$H_1^S = \nu a^\dagger a + \sum_i f_i \sigma_i^z + \sum_i (g_i^* a \sigma_i^+ + g_i a^\dagger \sigma_i^-), \quad (31)$$

$$H_2^S = \nu a^\dagger a + \sum_i f_i \sigma_i^z + \sum_i (g_i^* a^\dagger \sigma_i^+ + g_i a \sigma_i^-), \quad (32)$$

where the superscript S refers to the magnetic nature of the environment. Setting $f_i = f, \forall i$, and further choosing $f > 0$, the EOM in the Heisenberg picture are

$$\text{D}^S\text{-case:} \quad \dot{a} = i[H_1^S, a] = -i\nu a - i \sum_{i=1}^N g_i \sigma_i^-, \quad (33a)$$

$$\dot{\sigma}_i^- = i[H_1^S, \sigma_i^-] = -if \sigma_i^- + iag_i^* 2\sigma_i^z, \quad (33b)$$

$$\text{A}^S\text{-case:} \quad \dot{a} = i[H_2^S, a] = -i\nu a - i \sum_{i=1}^N g_i^* \sigma_i^+, \quad (34a)$$

$$\dot{\sigma}_i^+ = i[H_2^S, \sigma_i^+] = if \sigma_i^+ - iag_i 2\sigma_i^z, \quad (34b)$$

where we have related the index of the Hamiltonians $H_{1,2}^S$ with the dissipative (D^S) and amplifying (A^S) cases, respectively.

Despite Eqs. (33)-(34) have the same form as Eqs. (5)-(6) of the bosonic case, they cannot be solved exactly, due to the different algebra of the spin operators. However, restricting ourselves to physical situations where the operator $S^z \equiv \sum_{i=1}^N \sigma_i^z$ can be replaced by some reasonable expectation value $\langle S^z \rangle \equiv \frac{N}{2} \langle \sigma^z \rangle \equiv -\frac{N}{2} m$, (with $m > 0$, due to f being positive) we can rewrite the above EOM in the form

$$\text{D}^S\text{-case:} \quad \dot{a} = -i\nu a - i\Lambda \tilde{S}^-, \quad (35a)$$

$$\dot{\tilde{S}}^- = -if \tilde{S}^- - i\Lambda a \quad (35b)$$

$$\text{A}^S\text{-case:} \quad \dot{a} = -i\nu a - i\Lambda \tilde{S}^+, \quad (36a)$$

$$\dot{\tilde{S}}^+ = if \tilde{S}^+ + i\Lambda a, \quad (36b)$$

with $g = \sqrt{\sum_{i=1}^N |g_i|^2}$, $\Lambda = g\sqrt{2m}$, and

$$\tilde{S}^+ = \frac{1}{\Lambda} \sum_{i=1}^N g_i \sigma_i^+, \quad \tilde{S}^- = (\tilde{S}^+)^\dagger. \quad (37)$$

In fact, these equations can be derived from the Hamiltonians

$$\text{D}^S\text{-case:} \quad \nu a^\dagger a + f S^z + \Lambda(a \tilde{S}^+ + a^\dagger \tilde{S}^-), \quad (38)$$

$$\text{A}^S\text{-case:} \quad \nu a^\dagger a + f S^z + \Lambda(a^\dagger \tilde{S}^+ + a \tilde{S}^-), \quad (39)$$

upon further assuming that the commutation relations

$$[\tilde{S}^+, \tilde{S}^-] = -1, \quad [S^z, \tilde{S}^+] = \tilde{S}^+, \quad [S^z, \tilde{S}^-] = -\tilde{S}^-,$$

hold, meaning that the spin algebra is simplified into a bosonic one.

Notice that replacing the total spin operator $\sum_i \sigma_i^z$ with an expectation value $\langle S^z \rangle = \frac{N}{2} \langle \sigma^z \rangle$ we imply that the field f selects the same expectation value $\langle \sigma^z \rangle$ for every spin-1/2, in the spirit of the usual random phase approximation.

Once linearized, the EOM (35)-(36) can be solved as in the bosonic case, to get

$$\begin{aligned} a(t) &= e^{-iH_j^S t} a e^{iH_j^S t} = \left[\mu_j^S(t) a + \pi_j^S(t) \tilde{S}_j \right] e^{-i\omega_j^S t} \\ \tilde{S}_j(t) &= e^{-iH_j^S t} \tilde{S}_j e^{iH_j^S t} = \\ &\quad \left[(-)^j \pi_j^{S*}(t) a + \mu_j^{S*}(t) \tilde{S}_j \right] e^{-i\omega_j^S t}, \end{aligned} \quad (40)$$

where $\tilde{S}_1 = \tilde{S}^-$, $\tilde{S}_2 = \tilde{S}^+$,

$$\mu_j^S(t) = \cos(\Delta_j^S t) - i \frac{\delta_j^S}{\Delta_j^S} \sin(\Delta_j^S t), \quad (41a)$$

$$\pi_j^S(t) = (-)^j i \frac{\Lambda}{\Delta_j^S} \sin(\Delta_j^S t), \quad (41b)$$

with

$$\delta_j^S = \frac{1}{2} (\nu + (-)^j f), \quad (42a)$$

$$\omega_j^S = \frac{1}{2} (\nu - (-)^j f), \quad (42b)$$

$$(\Delta_j^S)^2 = |\delta_j^S|^2 - (-)^j \Lambda^2, \quad (42c)$$

and we have used $\pi_j^{S*}(t) = \pi_j^S(-t)$.

Whatever follows Eq. (12) in Sec. II can be easily retraced until the choice of the initial environmental state ρ_S appears into

$$\rho_A(t) = \text{Tr}_S \left[e^{-iH_j^S t} \rho_A \otimes \rho_S e^{iH_j^S t} \right] \equiv \mathcal{E}_j^S(\rho_A). \quad (43)$$

Assuming that S is initially prepared in a state at thermal equilibrium, we take

$$\rho_S = \frac{1}{1 + n_T^S} \left(\frac{n_T^S}{1 + n_T^S} \right)^{\tilde{S}^+ \tilde{S}^-}, \quad (44)$$

with $n_T^S \equiv \frac{N}{2} (1 - m)$.

Despite the formal analogy with Eq. (16), it is important to notice that the temperature-dependence of n_T^S , and hence that of the dynamical map, is generally different from what we get in the bosonic case, where the thermal number of photons is $n_T = (\exp\{\frac{\omega}{T}\} - 1)^{-1}$. We can, for example, suppose that the magnetic environment thermalizes with the thermal bath so that $\langle S^z \rangle = -\text{sgn}(f) S B_S(x) = -\frac{N}{2} \text{sgn}(f) m$, where $S = N/2$ and $B_S(x) = m$ is the Brillouin function

$$B_S(x) = \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S}x\right) - \frac{1}{2S} \coth\left(\frac{x}{2S}\right), \quad (45)$$

with $x = S|f|/T$. With this choice, it is $n_T^S \equiv S(1 - B_S(x))$ and the dependence on T of the bosonic model is only recovered when $T \rightarrow 0$, being $B_S(x) \rightarrow 1 - e^{-x}$ the low temperature limit of Eq. (45). Notice that, in order for the above representation to stay meaningful in the large- S limit, temperature must scale as $T \sim S$ so as to guarantee a finite x ; performing such large- S limit, the Brillouin function turns into the Langevin one

$$L(x) = \coth(x) - \frac{1}{x}, \quad (46)$$

which is indeed the classical limit of Eq. (45).

We observe that the approximations introduced for the spin system are consistent with our aim of finding an effective classical description for the environment: Indeed, once the total spin is guaranteed a constant value S , a classical-like behavior is expected for a spin-system when $S \gg 1$ [34, 39], and the bosonic expansion given by the Holstein-Primakoff transformation can be safely truncated at its lowest order $S^+ \sim b^\dagger$ (if $f > 0$, b^\dagger being a generic bosonic creation operator) [40].

We can now write the initial state $\rho_A \otimes \rho_S$ using the Glauber formula as in Eq. (17), with the spin displacement operator defined as $D_{\tilde{S}}(\gamma) = \exp\{\gamma \tilde{S}^+ - \gamma^* \tilde{S}^-\}$ due to the choice $f > 0$, and hence $\langle \sigma^z \rangle < 0$ (had we taken $f < 0$ it would be $D_{\tilde{S}}(\gamma) = \exp\{\gamma \tilde{S}^- - \gamma^* \tilde{S}^+\}$). Using Eqs. (40), the evolution of displacement operators is found to be

$$\begin{aligned} e^{-iH_j^S t} D_a^\dagger(\gamma') \otimes D_{\tilde{S}}^\dagger(\gamma'') e^{iH_j^S t} = \\ D_a^\dagger \left[\mu_j^{S*}(t) \gamma' - \pi_j^S(t) \gamma''^{*(*)j} \right] \otimes D_{\tilde{S}}^\dagger \left[(-)^j \pi_j^S(t) \gamma'^{*(*)j} - (-)^j \mu_j^{S*}(t) ((-)^j t) \gamma'' \right], \end{aligned} \quad (47)$$

and performing the partial trace of Eq. (43), being $\text{Tr}_S [D_{\tilde{S}}(\gamma)] = \pi \delta^{(2)}(\gamma)$, we get

$$\begin{aligned} \mathcal{E}_j^S(\rho_A) &= \int \frac{d^2 \gamma'}{\pi} \chi[\rho_A](\gamma') \chi[\rho_S] \left(\gamma'^{*(*)j} \frac{\pi_j^S(t)}{\mu_j^{S*}(t)^{*(*)j}} \right) D_a^\dagger \left(\frac{\gamma'}{\mu_j^S(t)} \right) \\ &\stackrel{\gamma' \rightarrow \gamma \mu_j^S(t)}{=} \int \frac{d^2 \gamma}{\pi} |\mu_j^S(t)|^2 \chi[\rho_A](\gamma \mu_j^S(t)) \chi[\rho_S](\gamma^{*(*)j} \pi_j^S(t)) D_a^\dagger(\gamma). \end{aligned} \quad (48)$$

Upon expanding the coefficients for short interaction time,

we find the same expressions of the bosonic case, since

$$\mu_j^S(t) \simeq 1 - i\delta_j t + O(t^2), \quad (49a)$$

$$\pi_j^S(t) \simeq (-)^j i\Lambda t + O(t^2), \quad (49b)$$

$$|\mu_j^S(t)|^2 \simeq 1 + O(t^2). \quad (49c)$$

We can now proceed as done in the previous section up to Eq. (21), thus obtaining that the dynamical map in the magnetic case does also correspond to a Gaussian noise channel. With the additional requirement of a random phase approximation, an effective Hamiltonian of the form of Eq. (29) can hence be written again, allowing us to conclude that the set of conditions sufficient to find an effective classical description is the same as in the bosonic model, the only difference being in the temperature dependence of the standard deviation σ^2 , due to the different definition of $n_{\mathcal{E}}^S$ in the magnetic case.

IV. LARGE- N ENVIRONMENT: DERIVING THE CLASSICAL FIELDS

In this section we take a more abstract view on the problem of what happens to the principal system A when its environment becomes macroscopic. Our aim is to understand whether the emergence of an effective, possibly non-autonomous, Hamiltonian $H_A^{\text{eff}}(\zeta(t))$ is a general feature of OQS with macroscopic environments. We also aim at further clarifying the meaning of the conditions (i)-(iii) given at the end of Sec. II, and the reasons why they seem to be utterly necessary in order to obtain an effective Hamiltonian description. Following suggestions from Refs. [28, 30, 34, 41], the main idea is to show that the emergence of $H_A^{\text{eff}}(\zeta)$ is related to the crossover from a quantum to a classical environment, possibly observed when the number of components becomes very large. In fact, were the environment described by a classical theory, its effects on the system would naturally be represented by the classical fields ζ .

Before introducing the general approach we are going to adopt, let us briefly recall some useful notions. A quantum description of a physical system is based on the introduction of (i) a Hilbert space \mathcal{H} , (ii) a Lie Algebra L , and (iii) a Hamiltonian H . The quantities $O(u) = \langle u|O|u \rangle$, i.e. the expectation values of Hermitian operators O acting on \mathcal{H} , are the (only) physical outputs of the theory, i.e. the experimentally accessible properties of the system. On the other hand, a classical model is defined by (i) a phase space \mathcal{M} , (ii) a Poisson bracket $\{\cdot, \cdot\}$, and (iii) a Hamiltonian h . Real functions defined on \mathcal{M} are the (only) physical outputs of the theory, in the same sense as above.

In what follows, we concentrate upon the model described by the Hamiltonian (1) and refer to the procedure of Ref. [34] to construct the classical theory that formally represents the large- N limit of any quantum theory with some global symmetry. In order to be used in the framework of OQS dynamics, such procedure needs being generalized, as we deal with the quantum theory of a bipartite system where just one of the two components, namely the environment, is intended to become macroscopic. However, due to the linear structure of the interactions entering Eq. (1), the procedure can be directly applied in what follows.

Let us first identify the global operators acting on \mathcal{H}_B in terms of which one can rewrite Eq. (1); these operators are guaranteed to exist when the total Hamiltonian is invariant under some group of global, unitary transformations acting on

\mathcal{H}_B only, whose knowledge is key for deductively construct the global operators in the general case. The Hamiltonian (1), however, is simple enough to allow us a visual recognition of them, as well of the conditions for their existence. Keeping in mind that our goal is that of introducing global operators that generate a physically meaningful Lie algebra, we notice that the coupling terms can be written as $a(a^\dagger)$ tensor-times some sum over k of operators acting on \mathcal{H}_B iff either $\lambda_{1k} = \lambda_{2k}$ or $\lambda_{1(2)k} = 0$, for all k . Taking one or the other of the above conditions true is quite equivalent, as far as the following construction is concerned: for the sake of clarity, and at variance with what done in Secs. II-III, we specifically choose $\lambda_{2k} = 0$ and set $\lambda_k \equiv \lambda_{1k}$ finite, for all k , meaning that we explicitly consider the dissipative case only. Further taking $\omega_k = \omega \forall k$, i.e. assuming a narrow environmental spectrum, as done in Secs. II-III, we can define the global operators

$$E \equiv \frac{1}{N} \sum_k^N b_k^\dagger b_k, \quad B \equiv \frac{1}{\sqrt{N}\Lambda^2} \sum_k^N \lambda_k b_k, \quad (50)$$

with $\Lambda^2 \equiv \sum_k^N |\lambda_k|^2$ as in Eq. (9). These operators generate, together with the identity, a Heisenberg algebra on \mathcal{H}_B ,

$$[B, B^\dagger] = \frac{1}{N}, \quad [B, E] = \frac{1}{N} B, \quad [B^\dagger, E] = -\frac{1}{N} B^\dagger. \quad (51)$$

The Hamiltonian (1) can then be written as

$$H = \nu a^\dagger a + N \left[\frac{\Lambda}{\sqrt{N}} (a^\dagger B + a B^\dagger) + \omega E \right] \quad (52)$$

$$\equiv \nu a^\dagger a + N h_N(a^\dagger, a; B^\dagger, B, E), \quad (53)$$

where the way N enters Eqs. (50-53) is specifically designed to recognize $\frac{1}{N}$ as the parameter to quantify quantumness of the environment B, and let all the operators, no matter whether acting on A, B, or A+B, independent on the number of environmental modes. Explicitly referring to the example given in Sec. IV of Ref. [34] we now introduce the set of antihermitian operators

$$\{L(\epsilon, \beta) \equiv iN(\epsilon E + \beta^* B + \beta B^\dagger)\}, \quad (54)$$

where $\beta \in \mathbb{C}$, with $|\beta| \propto \frac{1}{\sqrt{N}}$, while the coefficients $\epsilon \in \mathbb{R}$ do not depend on N . In the large- N limit, where terms which are bilinear in β and β^* can be neglected due to their dependence on N , it is $[L_1, L_2] = L_3$, with $L_i \equiv L_i(\epsilon_i, \beta_i)$, meaning that the set in Eq. (54) is a Lie Algebra, with $\beta_3 = i(\epsilon_1 \beta_2 - \epsilon_2 \beta_1)$ and $\epsilon_3 = 0$. This is the algebra whose recognition represents the first step towards the large- N limit of the quantum theory that describes B, i.e. such that one can write the part of the Hamiltonian that acts on \mathcal{H}_B as a polynomial function of its generators [34]. In fact, the Hamiltonian (52) is just a linear combination of $\{\mathbb{I}, E, B, B^\dagger\}_{\mathcal{H}_B}$. One can easily check that a 2×2 -matrix representation of the above algebra is

$$\left\{ \ell(\epsilon, \beta) \equiv i \begin{pmatrix} 0 & \beta^* \\ \epsilon & 0 \end{pmatrix} \right\}. \quad (55)$$

The choice of a representation that contains only either β or β^* is the simplest way to make the presence of non-commuting operators on \mathcal{H}_A in the renormalized Hamiltonian h_N harmless, as far as the following construction is concerned. Given that, for any pair of operators O and P , it is

$$e^{-P} O e^P = \sum_n \frac{(-1)^n}{n!} [P, [P, [\dots [P, O] \dots]]], \quad (56)$$

from the relations

$$\left[L, \begin{pmatrix} 1 \\ B \end{pmatrix} \right] = \ell^\dagger \begin{pmatrix} 1 \\ B \end{pmatrix} \text{ and } [L, (1 \ B^\dagger)] = (1 \ B^\dagger) \ell,$$

with $\ell^\dagger \equiv (\ell^*)^t$, it follows that the unitary operators $U(\phi, \psi) \equiv \exp\{L(\epsilon, \beta)\}$ obey

$$U^{-1} \begin{pmatrix} 1 \\ B \end{pmatrix} U = u(\phi, \psi) \begin{pmatrix} 1 \\ B \end{pmatrix} \quad (57)$$

and

$$U^{-1} (1 \ B^\dagger) U = (1 \ B^\dagger) u^\dagger(\phi, \psi), \quad (58)$$

with

$$u(\phi, \psi) \equiv \begin{pmatrix} 1 & 0 \\ \psi & \phi \end{pmatrix}, \quad (59)$$

where $\phi = \exp\{i\epsilon\}$ and $\psi = i\beta \int_0^1 \exp\{i\epsilon\tau\} d\tau$. The fact that the set (54) is a Lie algebra in the large- N limit reflects upon the unitary operators $U(\phi, \psi)$, in that they form a group in the same limit. This is indeed the group that defines, together with the arbitrary choice of a reference state $|0\rangle \in \mathcal{H}_B$, the normalized states $|u(\psi, \phi)\rangle \equiv U(\phi, \psi)|0\rangle$ such that $h_N(a^\dagger, a; B^\dagger, B, E)$ in Eq. (53) transforms into $h(a^\dagger, a; B^*(u), B(u), E(u))$ as N goes to infinity, with $O(u) \equiv \langle u|O|u\rangle$ the expectation value of any operator acting globally on B on the states $\{|u\rangle\}$. The reference state we choose is $|0\rangle = \Pi_k |0\rangle_k$, with $|0\rangle_k$ such that $b_k |0\rangle_k = 0$. This implies, given the separable structure of the operators $U(\phi, \psi)$, that the states $|u\rangle$ are tensor products of single-mode pure states. As a consequence, it is $\langle u|BB^\dagger|u\rangle = \langle u|\sum_{k'k} b_{k'} b_k^\dagger|u\rangle = \langle u|\sum_k b_k b_k^\dagger|u\rangle = E(u)$, which allows us to determine $B(u)$ and $E(u)$ via

$$\langle u| \begin{pmatrix} 1 \\ B \end{pmatrix} \otimes (1 \ B^\dagger) |u\rangle = \begin{pmatrix} 1 & B^*(u) \\ B(u) & E(u) \end{pmatrix}, \quad (60)$$

and finally obtain, by Eqs. (57-58) and again neglecting terms bilinear in β and β^* ,

$$\begin{aligned} \langle 0| u(\phi, \psi) \begin{pmatrix} 1 \\ B \end{pmatrix} \otimes (1 \ B^\dagger) u^\dagger(\phi, \psi) |0\rangle &= \\ &= \langle 0| \begin{pmatrix} 1 & \psi^* + \phi^* B^\dagger \\ \psi + \phi B & \psi\psi^* + \psi\phi^* B^\dagger + \psi^*\phi B + \phi\phi^* B B^\dagger \end{pmatrix} |0\rangle \\ &= \begin{pmatrix} 1 & \psi^* \\ \psi & 1 \end{pmatrix}. \end{aligned} \quad (61)$$

Recalling that $\psi \propto \beta$ and $|\beta| \propto \frac{1}{\sqrt{N}}$, the above result implies that the original Hamiltonian (53) formally transforms according to

$$H \xrightarrow{N \rightarrow \infty} H_A^{\text{eff}}(\zeta) = (\nu a^\dagger a + \omega) + \zeta^* a + \zeta a^\dagger. \quad (62)$$

where $(\zeta, \zeta^*) \in \mathbb{R}^2$ is proved [34] to be any point of a classical phase-space \mathcal{M}_B with canonical variables $q \equiv (\zeta + \zeta^*)/2$ and $p \equiv (\zeta - \zeta^*)/(2i)$. Notice that $|\zeta| \propto \Lambda/\sqrt{N}$, which is independent of N by definition.

Once Eq. (62) is obtained, we can maintain with confidence that the Hamiltonian (1), originally acting on $A+B$, formally transforms, as $N \rightarrow \infty$, into one that exclusively acts on A : however, the presence of the classical field ζ is the remnant of the underlying quantum interaction between A and the huge number of elementary constituents of B , namely the bosonic modes $\{b_k\}_{k=1}^N$. To this respect, notice that the Hilbert space $\mathcal{H}_B = \otimes_k \mathcal{H}_{b_k}$ is replaced by a two-dimensional classical phase-space, \mathcal{M}_B , implying an impressive reduction of dynamical variables. This reduction is the most striking consequence of the global symmetry that the quantum theory for B must exhibit in order to flow into a well defined classical theory when B is macroscopic. In our case, although we did not explicitly use it, the symmetry is that under permutation of the bosonic modes b_k , and that is why we have set $\omega_k = \omega \forall k$. In fact, one can easily check that this is an essential condition for the very same definition of global operators obeying commutation rules of the form (51), which on their turn are necessary to proceed to the definition of the Lie Algebra, and all the rest.

At this point, we notice that $\omega_k = \omega \forall k$ is just the “flat environmental energy-spectrum condition” (i), discussed at the end of Sec. II. In fact, it immediately strikes that the effective Hamiltonian in Eq. (62) has the same structure of that in Eq. (29), given that the latter refers to an interaction picture that hides the environmental frequency ω . On the other hand, it is somehow puzzling that time does not enter the above construction, which leave us clueless, so far, concerning the relation $\zeta \rightarrow \zeta(t)e^{-i\omega_\zeta t}$.

Looking for the possible origin of a time-dependence in the classical field ζ , we reckon that the results of this section imply the following. Suppose there exists another macroscopic system T that is not coupled with A , and interacts with B in such a way that the above global symmetry is preserved: the presence of T manifests itself in terms of some parameter τ (think about time and/or temperature, for instance) upon which ζ depends, according to the rule $\zeta = \zeta(\tau)$ provided by the classical theory describing $B+T$. This dependence can be safely imported into the effective description of A via $\zeta \rightarrow \zeta(\tau)$ in $H_A^{\text{eff}}(\zeta)$, Eq. (62), as far as the direct interaction between A and T can be neglected, at least on the time scales one is interested in.

Finally, we notice that the detuning $\nu - \omega$ does not play any role in this section, which brings us back to Eq. (30) and the possible relation between the large- N condition here enforced and the short-time approximation previously adopted.

V. CONCLUSIONS

In this paper, we have addressed the dynamics of a bosonic system coupled to either a bosonic or a magnetic environment. In particular, we have discussed the conditions under which the dynamics of the system may be described in terms of the effective interaction with a classical fluctuating field.

Our results show that for both kinds of environments an effective, time-dependent, Hamiltonian description may be obtained for short interaction time and environments with a flat energy spectrum at thermal equilibrium. The corresponding dynamics is described by a Gaussian noise channel independently of the kind of environment, their magnetic or bosonic nature entering only the form of the noise variance. As far as the energy spectrum is flat, this effective description is valid at all temperatures and independently on the nature of the interaction between the system and its environment.

Moreover, exploiting a general treatment based on the large- N limit of the environment, we have clarified the origin and the meaning of the short-time and flat-environmental-spectrum conditions. In fact, we find that a flat spectrum is needed for a global symmetry to emerge and characterize the

environment, which is a necessary ingredient for the environment to be described by a small number of macroscopic variables. On the other hand, the large energy scale implied by whatever coupling with a macroscopic environment limits any effective description to short times only.

Overall, our results indicate that quantum environments may be described by classical fields whenever global symmetries allows one to define environmental operators that remain well defined when increasing the spatial size of the environment. This is a quite general criterion that may serve as a guideline for further analysis, e.g. for fermionic principal systems and/or hybrid environments.

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